Law of large numbers for auto-inhibited Hawkes processes

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A Hawkes process is:

- random
- temporal
- point (jump process)

Independence

A well-known random process is the Poisson process.

- Independent interarrivals
- Same law (Exponential law)

Modelizes the arrival of customers in a shop.

Application of Hawkes process

First, Hawkes process are studied for earthquakes.

- Dependent of what happened before, with aftershocks.
- Time-limited dependence
- Randomness

Other modelizations:

- Social network
- Finance
- Neurons

Definition

Let $\lambda > 0$ and $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function.
A Hawkes process N^h of initial intensity λ is a self-influencing point process whose intensity is given at each time $t \geq 0$ by:

$$\Lambda^h : t \in (0, +\infty) \mapsto = \left(\lambda + \sum_{i \geq 1} h(t - U_i) \right)^+ \\ = \left(\lambda + \int_{(-\infty,t)} h(t-u) N^h(du) \right)^+ ,$$

where $(U_i)_i$ are the jumps of N^h.

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where $(U_i)_i$ are the jumps of N^h.
More generally:

$$\Lambda^h(t) = \Phi\left(\int_{(-\infty,t)} h(t-u) N^h(du)\right)$$

where $\Phi : \mathbb{R} \to \mathbb{R}^+$.

Construction

Proposition

Let Q be a $(\mathcal{F}_t)_{t\geq 0}$ - two-dimensional Poisson measure on $(0,+\infty)\times(0,+\infty)$ with unit intensity. We consider the equation

$$\begin{cases} \Lambda^h(t) = \left(\lambda + \int_{(-\infty,t)} h(t-u)N^h(du)\right)^+, \ u>0, \\ N^h = \int_{(0,+\infty)\times(0,+\infty)} \delta_u \mathbb{1}_{\theta\leq\Lambda^h(u)} Q(du,d\theta) \end{cases} \tag{1}$$

Then, under some hypothesis, there exists a solution and this solution is a Hawkes process.

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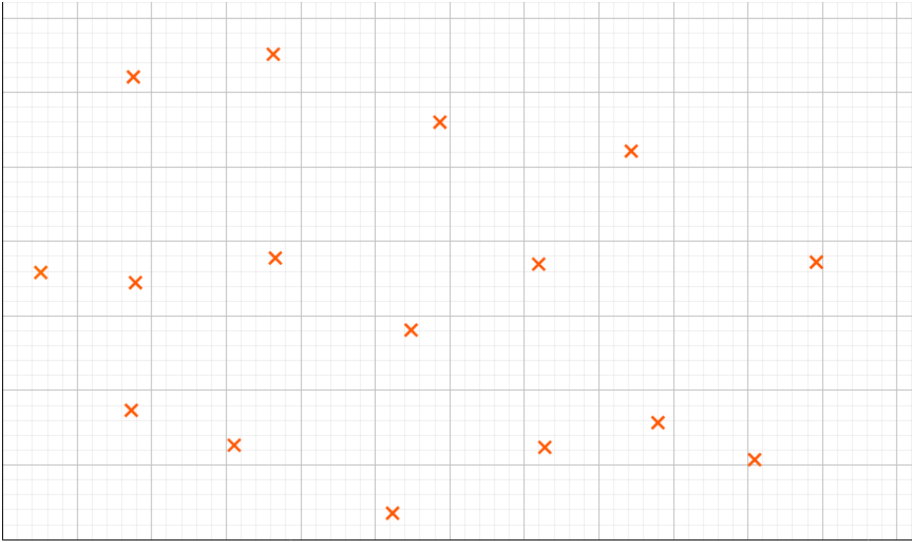
Then, under some hypothesis, there exists a solution and this solution is a Hawkes process.

We can see N^h as : $N^h = \sum_{i\geq 1}\delta_{U_i}$ where U_i are the jumps.

Construction of a process with a deterministic function



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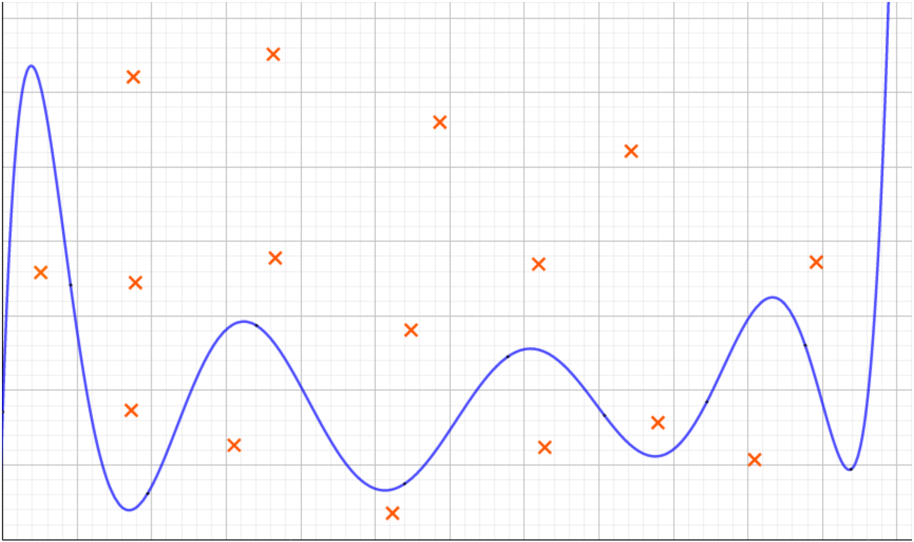

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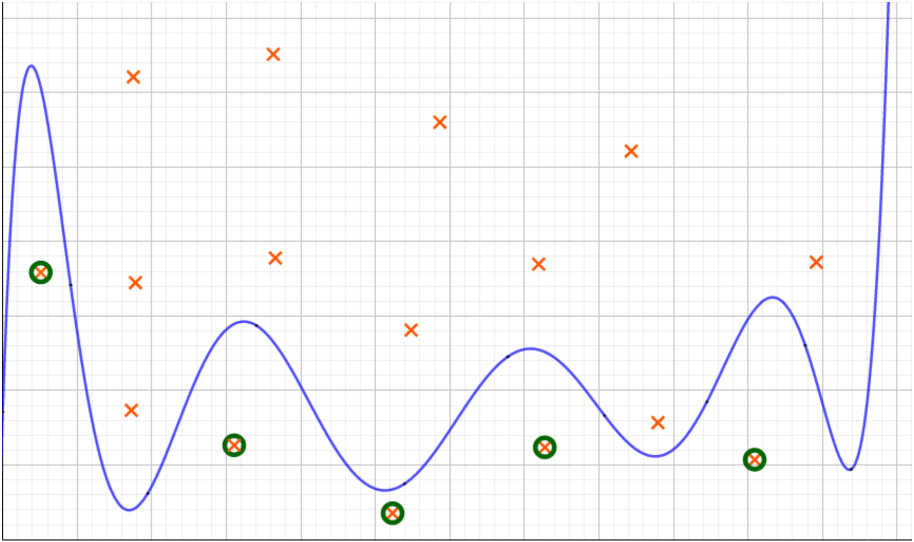

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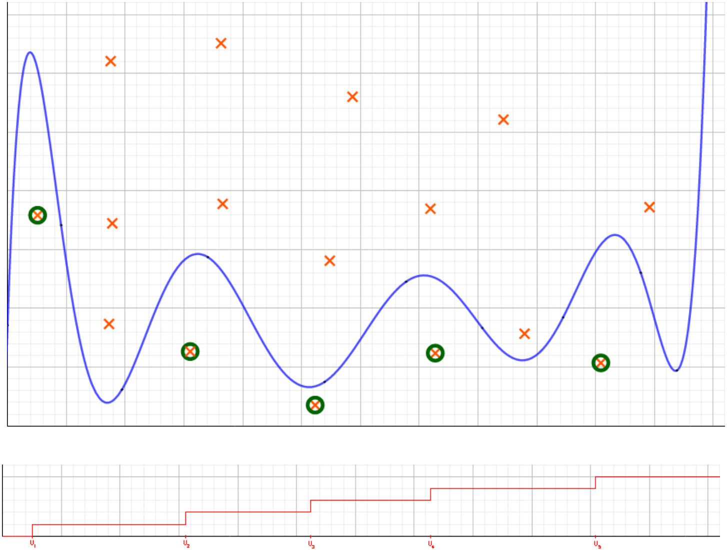

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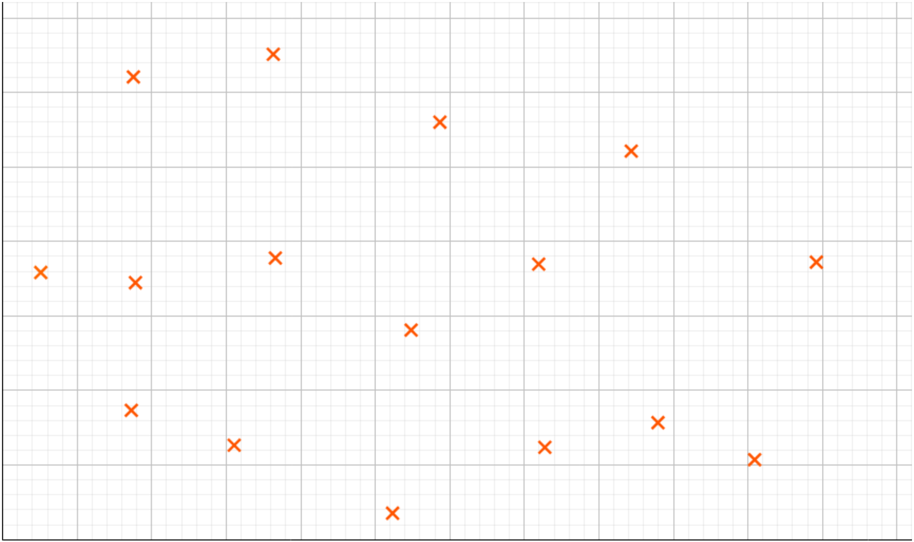

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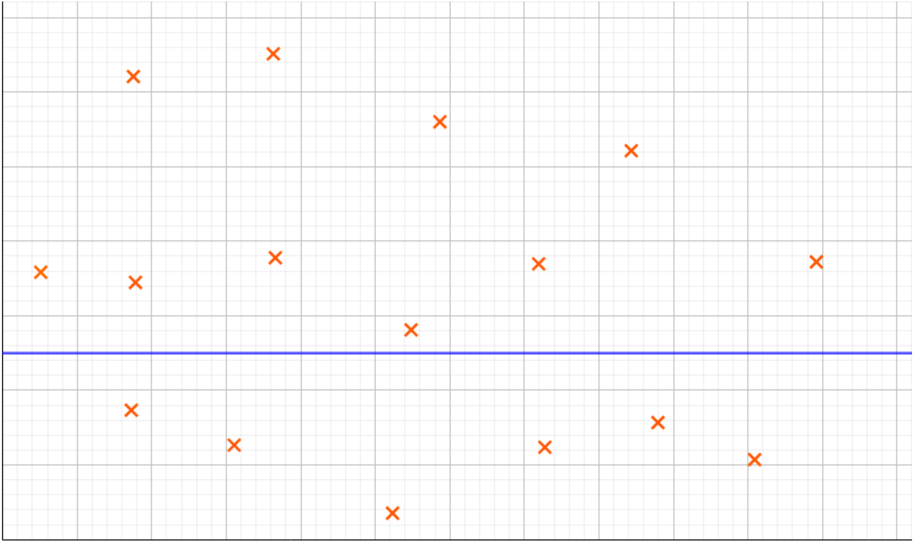

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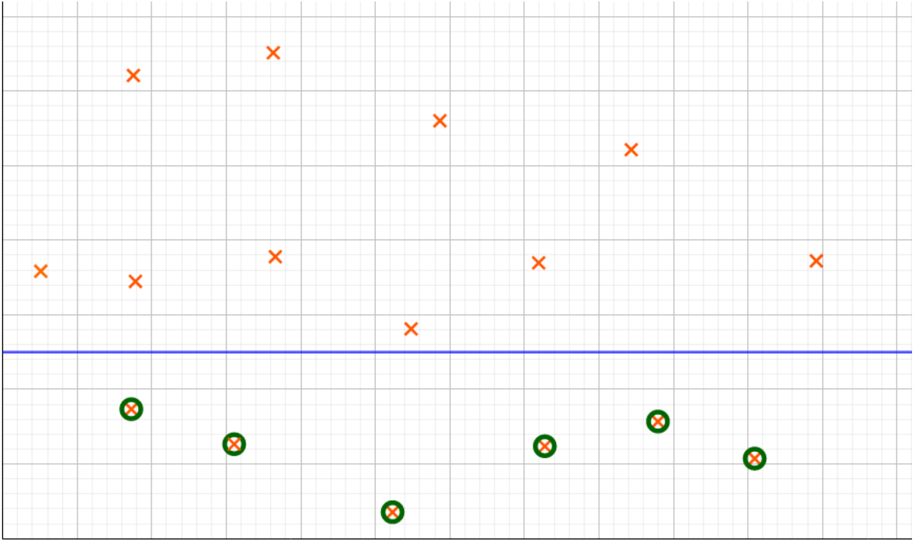

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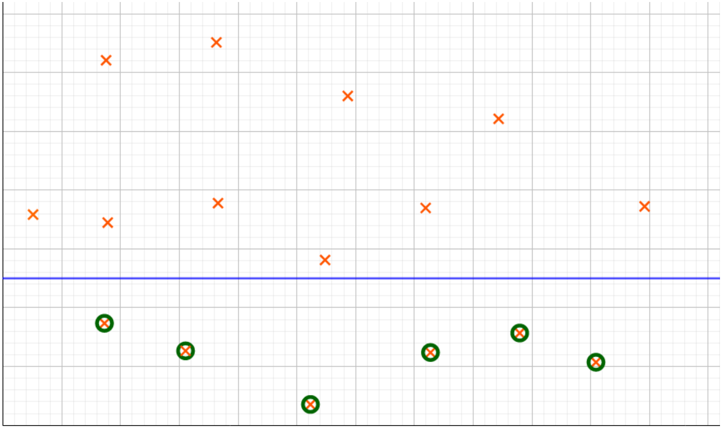
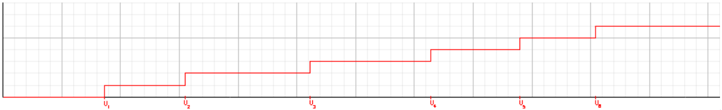




Construction of a Hawkes process

Example

$\lambda = 1$

$h = \mathbb{1}_{[0,0.5]} - \mathbb{1}_{(0.5,1]}$

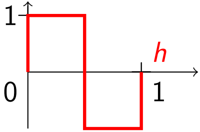
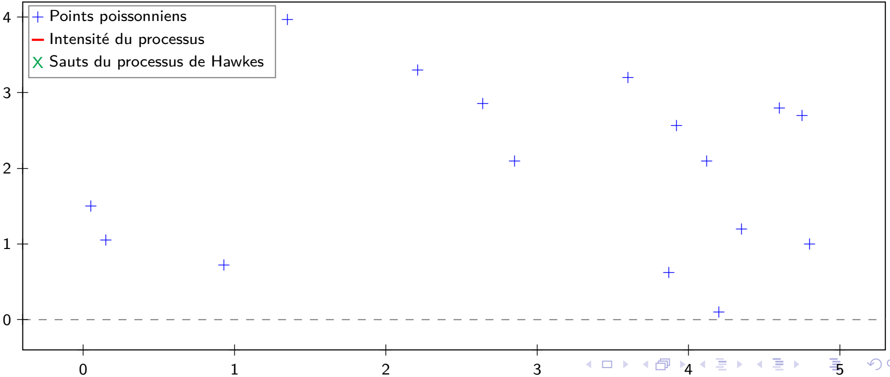


Figure: function h



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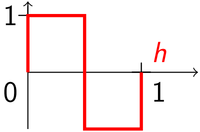

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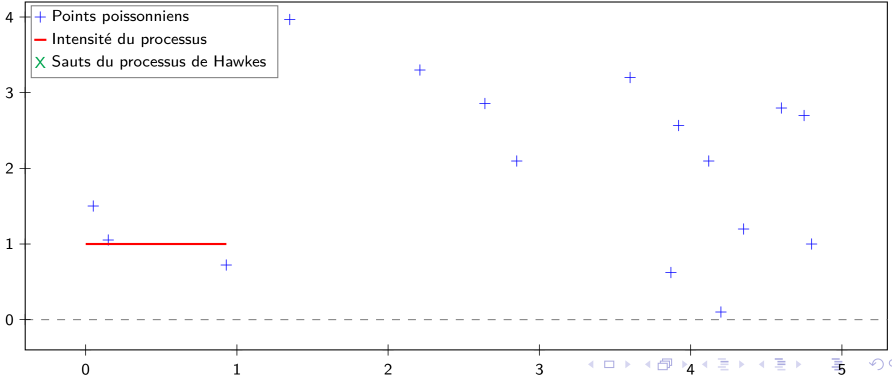

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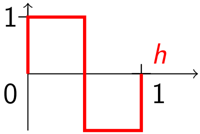

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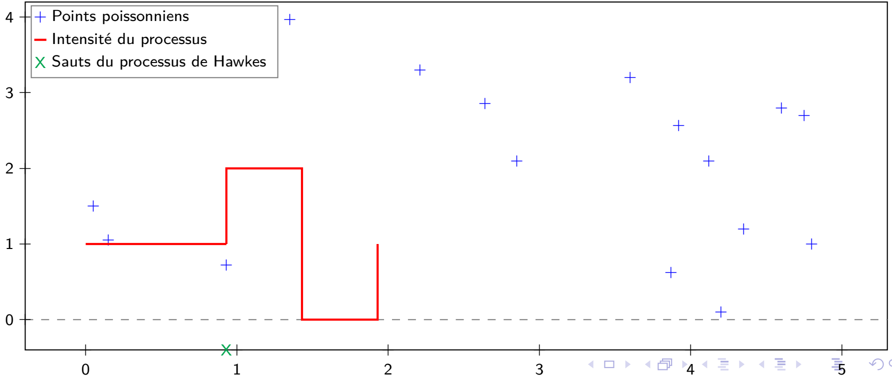

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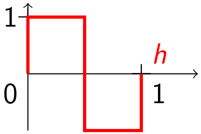

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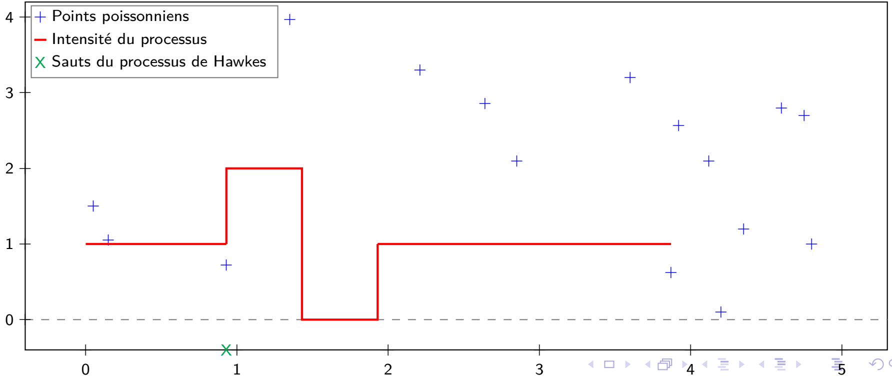

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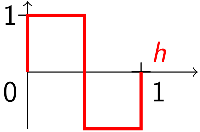

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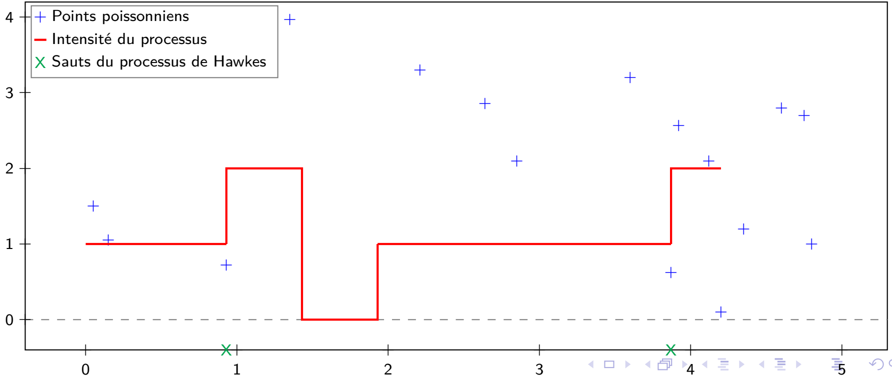

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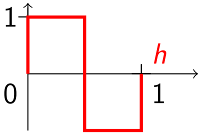

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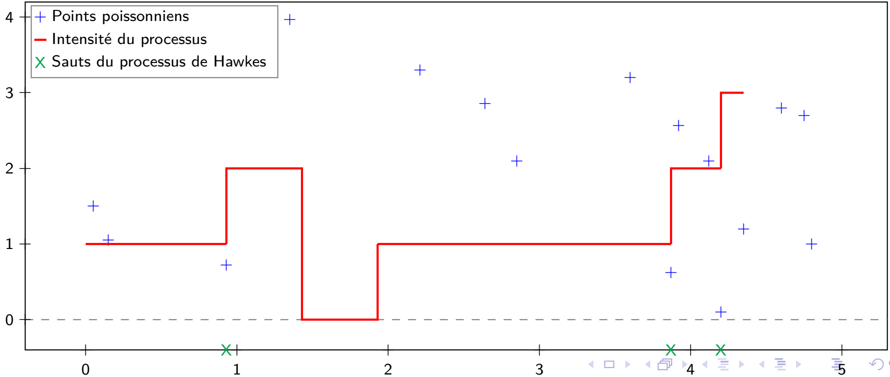

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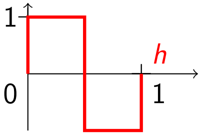

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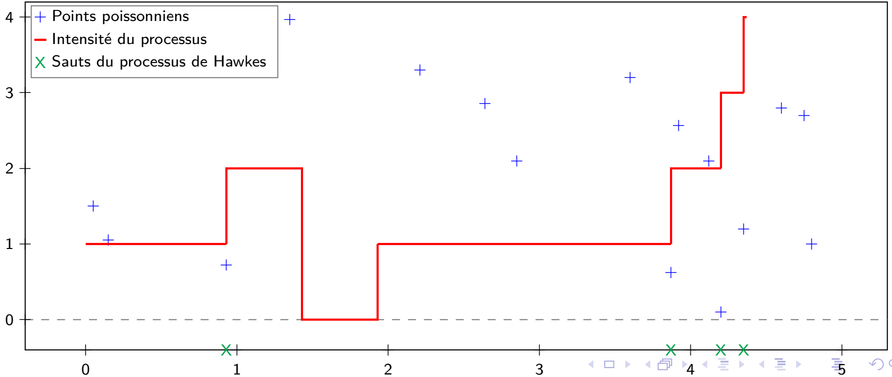

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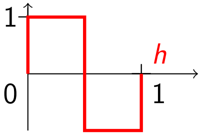

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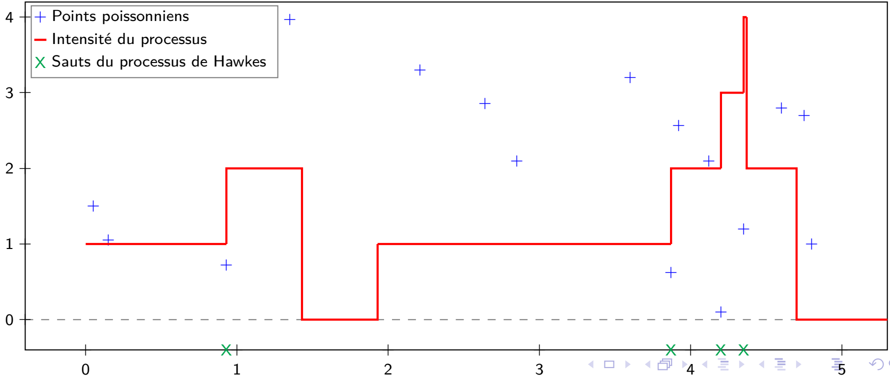

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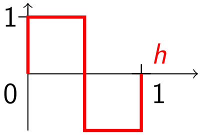

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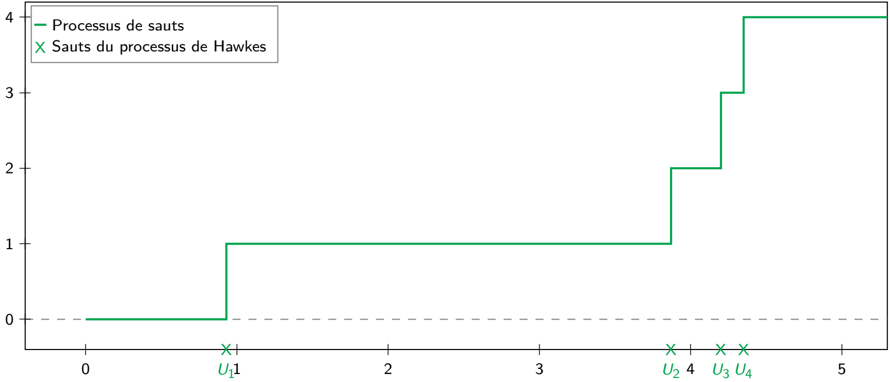

Proposition (Majoration of N^h)

The existence and the construction are similar for N^{h^+}.
We can construct N^h and N^{h^+} with the same Poisson-measure Q, and we have: $N^h \leq N^{h^+}$.

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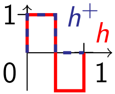
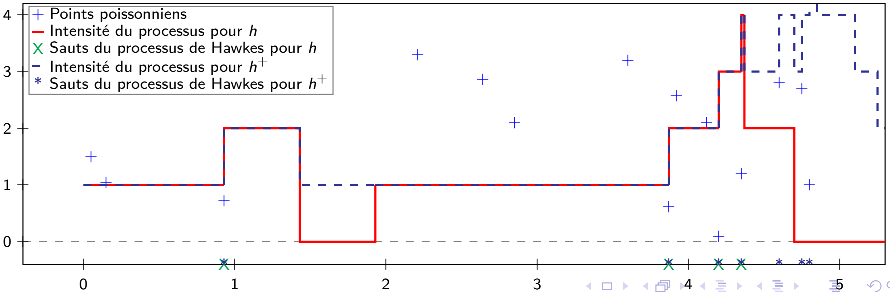


Figure: functions h and h^+



Lemma (Minoration of N^{h^+})

Let h be a function with a compact support $[0, L(h)]$. Let $\lambda > 0$ be the initial intensity.
We define $g = -\lambda \mathbb{1}_{[0,L(h)]}$. We construct N^h and N^g with the same Poisson-measure Q.
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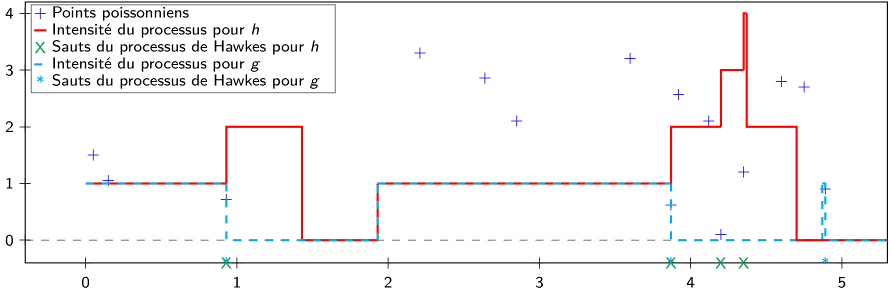

Law of large numbers

Remark (Law of large numbers)

(X_i) i.i.d such that $\mathbb{E}[X_1] < +\infty$ then $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1]$.

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Proposition (Law of large numbers for Poisson process)

Let $\mathcal{R}$ be a Poisson process of parameter λ. Then, we have:

$$\frac{\mathcal{R}([0,t])}{t} = \frac{\mathcal{R}_t}{t} \xrightarrow[t\to\infty]{a.s.} \lambda.$$

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What about $\dfrac{N^h([0,t])}{t} = \dfrac{N_t^h}{t}$ if $h \leq 0$?

Probabilists love independence

Idea

We have a function h with *compact* support $[0, L(h)]$
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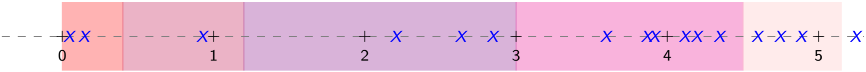

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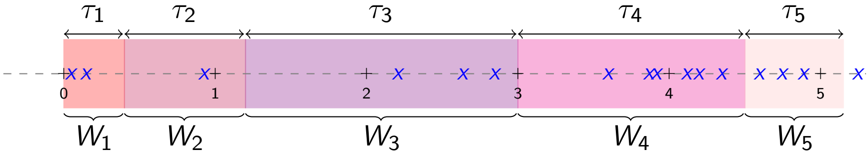

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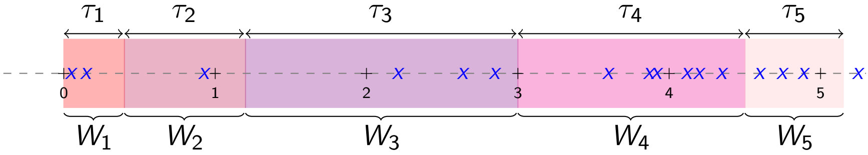


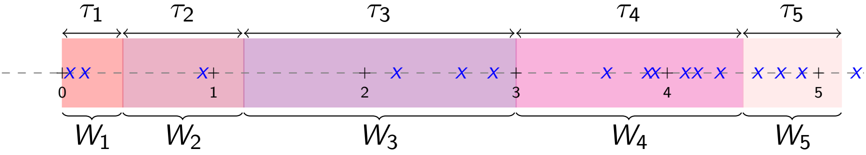
Intensity

$\Lambda^h(t) = \left(\lambda + \sum_{i\geq 1} h(t - U_i)\right)^+$
When do we have:
$\sum_{i\geq 1} h(t - U_i) = 0$?

Definition of τ and W

$$\tau_1 = \inf\{t > U_1^1, N^h((t - L(h), t]) = 0\}$$

$$W_1 = N^h([0, \tau_1]).$$



Definition of τ and W

$\tau_1 = \inf\{t > U_1^1, N^h((t - L(h), t]) = 0\}$
$W_1 = N^h([0, \tau_1])$.
$S_0 = 0,\ S_{i+1} = S_i + \tau_{i+1}$,
$\tau_{i+1} = \inf\{t > U_1^{i+1} - S_i, N^h((t + S_i - L(h), t + S_i]) = 0\}$,
$W_{i+1} = N^h([S_i, S_{i+1}])$

Renewal process

Definition

Let $(\tau_i)_i$ *i.i.d., non-negative, and* $S_i = \sum_{j=1}^{i} \tau_j$.
$(S_k)_k$ *is named renewal process and we consider the counting process associated:*

$$\forall t \in \mathbb{R}^+, M_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{S_n \leq t}.$$

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Theorem (Convergence theorem for renewal process)

If $\mathbb{P}(\tau_1 < \infty) = 1$ *then:*

$$M_t \xrightarrow[t \to +\infty]{a.s.} \infty.$$

Moreover, if τ *has a finite mean, we have:*

$$\frac{M_t}{t} \xrightarrow[t \to +\infty]{a.s.} \frac{1}{\mathbb{E}[\tau_1]}.$$

Law of large numbers for Hawkes processes

Proposition (Law of large numbers)

Let h be a negative function, with a support includes in $[0, L(h)]$. Then we have:

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}.$$

Law of large numbers for Hawkes processes

Idea

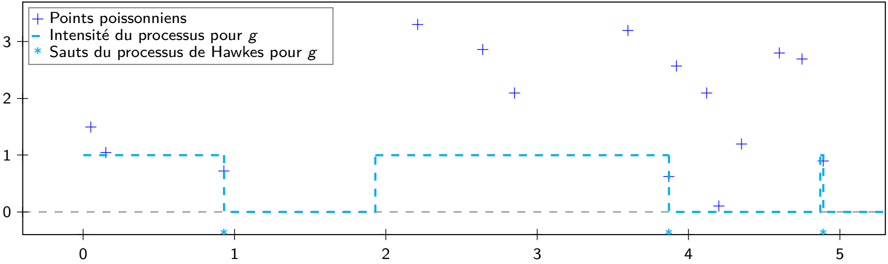
- $\frac{N_t^h}{t} \simeq \frac{1}{t} \sum_{i \geq 1} W_i \mathbb{1}_{S_i \leq t}$.
- We apply the convergence theorem for $M_t = \sum_{i \geq 1} \mathbb{1}_{S_i \leq t}$. We know: $M_t \xrightarrow[t \to +\infty]{a.s.} \infty$ and $\frac{M_t}{t} \xrightarrow[t \to +\infty]{a.s.} \frac{1}{\mathbb{E}[\tau_1]}$.
- We decompose $\frac{1}{t} \sum_{i \geq 1} W_i \mathbb{1}_{S_i \leq t}$:

$$\frac{1}{t} \sum_{i \geq 1} W_i \mathbb{1}_{S_i \leq t} = \frac{1}{t} \sum_{i=1}^{M_t} W_i = \frac{M_t}{t} \frac{1}{M_t} \sum_{i=1}^{M_t} W_i$$

- We have: $\frac{M_t}{t} \xrightarrow[t \to \infty]{a.s.} \frac{1}{\mathbb{E}[\tau_1]}$ and by LLN: $\frac{1}{M_t} \sum_{i=1}^{M_t} W_i \xrightarrow[t \to \infty]{a.s.} \mathbb{E}[W_1]$.
- Then $\frac{N_t^h}{t} \xrightarrow[t \to \infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$.

Application

Let $g = -\lambda \mathbb{1}_{[0,A]}$.



Here: $W_1 = 1$ a.s. and $\tau_1 \sim A + \mathcal{E}(\lambda)$.

Then $\dfrac{N_t^g}{t} \xrightarrow[t\to\infty]{a.s.} \dfrac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]} = \dfrac{1}{A + \lambda^{-1}} = \dfrac{\lambda}{\lambda A + 1}$.

Reminder

Let h be a negative function, with compact support $[0, L(h)]$.

- Almost surely, for each $t \in \mathbb{R}^+$, $N_t^g \leq N_t^h$, for $g = -\lambda \mathbb{1}_{[0,L(h)]}$.
- Almost surely, $N^h \leq N^{h^+}$. Here, $h^+ = 0$, so N^{h^+} is a Poisson process.

Consequence

We have:

$$\frac{\lambda}{\lambda L(h) + 1} \leq \lim_{t \to \infty} \frac{N_t^h}{t} \leq \lambda \quad a.s.$$

Conclusion

We have:

Proposition (Law of large numbers)

Let h be a negative function, with a support includes in $[0, L(h)]$*. Then we have:*

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}.$$

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$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}.$$

In fact, we have:

Proposition (Law of large numbers)

*Let h be **any** function, with a support includes in $[0, L(h)]$. Then we have:*

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}.$$

Conclusion

- For any h with compact support, we also have a Limit Central Theorem.
- For negative h and under some assumptions, there exists a Large Deviations Principle.

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